Logical models for trial and error mathematics:

Dialectical and quasidialectical systems

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Joint work

A1: J. Amidei, D. Pianigiani, L. San Mauro, G. Simi, and A. Sorbi, **Trial and error mathematics I: Dialectical and quasidialectical systems**, *Review of Symbolic Logic*, 9(2), 299–324, 2016

A2: J. Amidei, D. Pianigiani, L. San Mauro, and A. Sorbi, **Trial and error mathematics II: Dialectical sets and quasidialectical sets, their degrees, and their distribution within the class of limit sets**, to appear in *Review of Symbolic Logic*

A3: J. Amidei, U. Andrews, D. Pianigiani, L. San Mauro, and A. Sorbi, **Quasidialectical systems and the completions of first-order theories**, in preparation, 2016

Opening

Informal vs Formal (1)

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Of course quite the same can be said about the question: What is a mathematical proof?

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As a matter of fact, a growing number of philosophers of mathematics argue that FT does not hold (e.g., consider the practical turn advocated by the so-called Philosophy of Mathematical Practice).

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Lakatos, Proof and Refutations (1976)

The subject matter of metamathematics is an abstraction of mathematics in which mathematical theories are replaced by formal systems [...]. [But] there are problems which fall outside the range of metamathematical abstractions. Among these are problems relating to informal mathematics and to its growth, and all problems relating to the situational logic of mathematical problem-solving. [...] Formalist disconnects the history of mathematics from the philosophy of mathematics, since, according to formalist concept of mathematics, there is no history of mathematics proof. [...] According to formalist, mathematics is identical with formalised mathematics.

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- Visual and diagrammatic reasoning (Manders, 1995);
- Fruitfulness of mathematical concepts (Tappenden, 2008);
- Trial and error processes (Magari, 1974 - A., 2015).

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Larvor, How to think about informal proofs, 2012

The philosophy of mathematical practice prides itself on paying attention to the proofs that mathematicians offer each other, rather than the abstract models of proofs studied in formal logic. [...] However, it remains somewhat under-theorised. Among other things, the field lacks an explication of 'informal proof' as it appears in expressions such as 'the informal proofs that mathematicians actually read and write'. Without this, it is difficult to explain how studies of practice might diagnose and overcome the short-comings of those approaches that take formal logic to supply an adequate account of mathematical inference.

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1. the general idea that real mathematical theories differ from formal systems;
2. and yet the fact that most of this difference is presented by means of referring to some very non-logical aspects of mathematics, and these presentations sometimes "amounts to more than a mere advertisement for a future theory of informal [. . .] provability" (Leitgeb, 2012).

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We aim thus to describe a logical model – introduced in (Magari, 1974) – that extend formal systems in a way that is much more sensible to certain aspects of real mathematical theories that are classically neglected.

Of course this does not mean that these models are generally safe or immune from the kind of arguments anti-formalists typically present.

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3. We prove several mathematical results concerning both systems. In particular we compare the two systems with respect to both their informational content and the class of sets they "represent";

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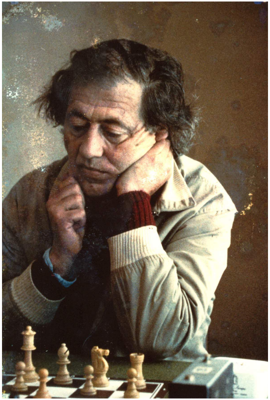
4. Finally, we discuss how to equip these systems with additional rules which mimic that of classical logic.

Dialectical systems

Roberto Magari

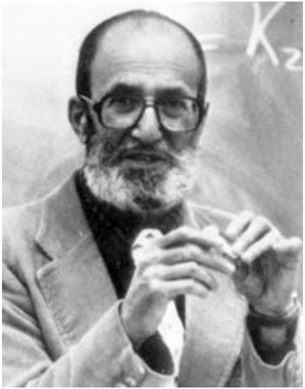


Roberto Magari (1934-1994)

The needs that are pushing one to modify the theory taken as metamathematics have the same nature as those pushing physicists or any natural scientist.

- R. Magari. *Su certe teorie non enumerabili.* Ann. Mat. Pura Appl. (4), XCVIII:119-152, 1974.

Trial and error, experimentation, guesswork



Paul Halmos (1916-2006)

Mathematics is not a deductive science – that's a cliché. When you try to prove a theorem, you don't just list the hypotheses, and then start to reason. What you do is trial and error, experimentation, guesswork.

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Magari's original idea, then, was precisely to provide an extension of formal systems that would be able to capture this dynamic feature.

An informal description of a dialectical system

The basic ingredients of a dialectical system are a number c, called a contradiction; a deduction operator H that tells us how to derive consequences from a finite set A of assumptions; a proposing function, i.e. a computable function f that proposes axioms, to be accepted or rejected as provisional theses of the system.

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For sake of simplicity, in what follows we will always denote $f(i)$ by f_i.

If up to a given stage we have accepted the axioms $f(i_1), \ldots, f(i_n)$, with $i_1 < \cdots < i_n$, and at this stage we see that we can derive c from $f(i_1), \ldots, f(i_m)$, for a least $m \leq n$, then we temporarily reject $f(i_m)$, still accept $f(i_1), \ldots, f(i_{m-1})$, and we are willing to add (perhaps again) $f(i_m + 1)$ to our working assumptions; on the other hand, if we see that c does not arise, then we are willing to add $f(i_n + 1)$ to our working assumptions.

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Recall that an enumeration operator H is a c.e. set, and

$$H(X) = \{x : \langle x, D\rangle \in H \;\&\; D \subseteq X\}$$

where D is a finite set. We often refer to computable approximations $\{H_s\}$ to a given enumeration operator H.

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   2. there is a least $z \leq m$, such that $c \in H_s(\{f_i \in A_s : i \leq z\} \cup \{f_m\})$, then let
   $$A_{s+1} = H_{s+1}(\{f_j \in A_s : j < z\}),$$
   and propose f_{z+1}.

Case 1: No contradiction: Just go on!

f_0			f_3		f_5		f_m
0	1	2	3	4	5	$\cdots\cdots\cdots$	m

Configuration at s

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Configuration at $s+1$, via clause (1)

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Case 2. Has c appeared? Then discard and try again!

f_0			f_3		f_5		f_z			f_m
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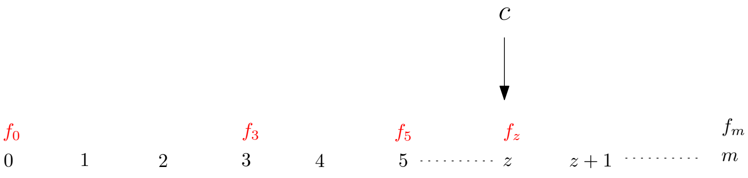

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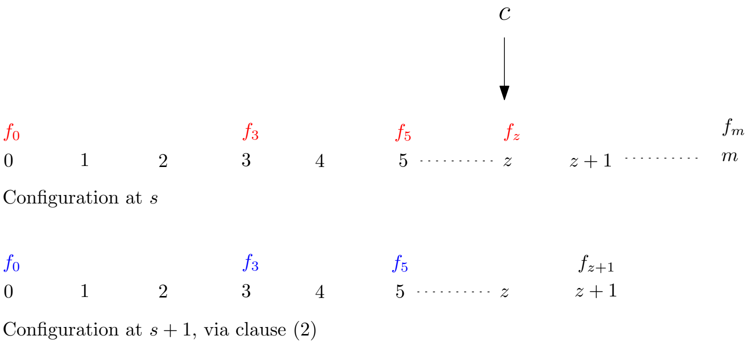

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Final theses and dialectical set

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For all dialectical systems $d = \langle H, f, c \rangle$, it can be shown that A_d is invariant with respect to how we approximate H.

Dialectical sets and the arithmetical hierarchy

In order to characterize the informational content of dialectical systems (and compare it with that of formal systems), we shall ask the following natural question:

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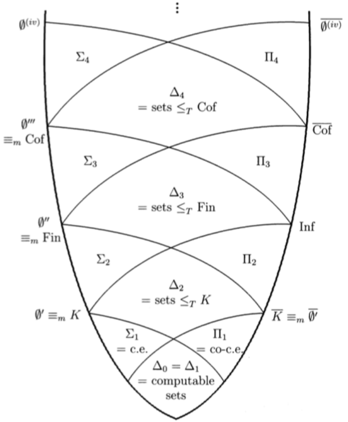
How dialectical systems are distributed within the arithmetical hierarchy?



Dialectical sets are Δ^0_2 (1)

First, notice that, from the definition of final theses, it follows immediately that every dialectical set is Σ^0_2. Indeed

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Theorem. (Magari)

For every dialectical system d, the corresponding dialectical set A_d is Δ^0_2.

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Lemma

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Therefore, whether some axiom f_x belongs to A_d is something that is fully determined by the behaviour of only those axioms that are proposed before f_x.

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Notice that H is c.e. and is an algebraic closure operator. Take $d = \langle H, f, c\rangle$, where f is the identity. It is easy to see that $A = A_d$.

Every c.e. dialectical set is computable

Nonetheless not all Δ^0_2 sets are dialectical!

Theorem (Magari)

No noncomputable c.e. set is dialectical.

Proof

Let A be a c.e. set. Consider the following function

$$\varphi(f_x) = \begin{cases} 1 & f_x \in A_d \\ 0 & c \in H(A_d \cap \{f_y : y < x\} \cup \{f_x\}) \end{cases}$$

By Magari's lemma, we have that φ is the characteristic function of A_d.

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Nonetheless not all Δ^0_2 sets are dialectical!

Theorem (Magari)

No noncomputable c.e. set is dialectical.

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Dialectical degrees

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Yet, before answering to this question, let us introduce the second class of systems we focus on.

Quasidialectical systems

From dialectical to quasidialectical systems

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Dialectical systems seem to be unfit for such cases, since each contradiction imposes to discard the axiom, and no substitution, or refinement, is considered.

The case of geometry

Euclidean geometry

1. Any two points can be joined by a straight line.
2. Any straight line segment can be extended indefinitely in a straight line.
3. Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.
4. All right angles are congruent.
5. The parallel postulate: Through a point not on a given straight line, one and only one line can be drawn that never meets the given line.

The case of geometry

Spherical geometry

1. Any two points can be joined by a straight line.
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The case of geometry

Hyperbolic geometry

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2. Any straight line segment can be extended indefinitely in a straight line.
3. Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.
4. All right angles are congruent.
5. The parallel postulate: There are NO parallel lines.Through a point not on a given straight line, infinitely many lines can be drawn that never meet the given line.

From dialectical to quasidialectical systems, continued

Thus, we propose to modify Magari's original definition by introducing some new systems (that we call quasidialectical systems) apt to accommodate this idea of revision.

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Then, we will compare them to dialectical systems in terms of their expressiveness and information content, thus verifying whether such a notion of revision can be already embedded in Magari's systems.

From dialectical to quasidialectical systems, continued

Quasidialectical systems extend standard dialectical systems with two additional symbols: c^- and f^-. Roughly, the role of f^- is that of replacing a certain axiom, that has produced some kind of problem, formally encoded by c^-, with another axiom. Thus, while c represents the mathematical contradiction, c^- corresponds to a large variety of possible problems that might lead a mathematician to replace an axiom.

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At the very high level of generality in which our presentation is pursued, the specific nature of these kind of problems is disregarded. That is, we do not want to commit ourselves to the specific semantic status of c^-.

From dialectical to quasdialectical systems, continued

On the contrary, our aim is to keep the intended meaning of c^- vague enough to incorporate a wide class of problems. These problems do not necessarily pertain to the formal side of the mathematical practice. Indeed, due to the generality of our proposal, they might include problems related to that kind of informal *desiderata* one can expect from an axiom, such as fruitfulness, or simplicity – or even psychological and aesthetic features, these latter being fully admissible as long as they can represent some reason to replace a given axiom.

Quasidialectical System: Definition

Definition (A1)

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2. $c^- \in \omega$;
3. f^- is a total computable function and $c^- \notin range(f^-)$;
4. f^- is acyclic, i.e., for every x, the f^--orbit of x is infinite.

Why f^- has to be acyclic?

We want to restrict ourselves to systems in which the operation of replacement is somewhat always enriching, in the following sense. Suppose we find some axiom unsatisfactory (again, this could be for a plenty of different reasons). Then we replace it. Later on, some problem occurs with this latter axiom, and thus we replace it too, with a third one. Now, if one aims at harmonizing the definition of f^- with some informal idea of "trial and error", in which knowledge is obtained through a process of refining subsequent proposals, then it is natural to ask that this third axiom is different from the first one we already replaced. Being acyclic just generalizes this intuition.

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We also have a computable function $r_s(i)$ where for each i, $r_s(i) = \langle\rangle$, or $r_s(i) = \langle f_i, f^-(f_i), \ldots, (f^-)^{n_i}\rangle$ for some n_i: we call $r_s(i)$ the stack at i, at stage s; by $\rho_s(i)$ we denote the top of the stack $r_s(i)$, i.e.

$$\rho_s(i) = \begin{cases} \emptyset, & r_s(i) = \emptyset; \\ (f^-)^{n_1}(f_i), & \text{if } r_s(i) = \langle f_i, f^-(f_i), \ldots, (f^-)^{n_i}\rangle \text{ for some } i; \end{cases}$$

there is a greatest m such that $r_s(m) \neq \langle\rangle$, and in this case $r_s(m) = \langle f_m\rangle$; and we denote by $L_s(i) = \{\rho_s(j) : j < i \text{ and } r_s(y) \neq \langle\rangle\}$.

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   1. $c, c^- \notin H(L_s(m) \cup \{\rho_s(m)\})$: define $r_{s+1}(m+1) = \langle f_{m+1} \rangle$, and $r_{s+1}(i) = r_s(i)$ otherwise, and let $A_{s+1} = H_{s+1}(L_{s+1}(m))$;

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   2. there is a least $z \leq m$ such that, $\rho_s(z) \neq \emptyset$, and $c \in H(L_s(z) \cup \{\rho_s(z)\})$: then let $r_{s+1}(i) = r_s(i)$, if $i < z$, $r_{s+1}(z+1) = \langle f_{z+1} \rangle$, empty all other stacks, and let $A_{s+1} = H_{s+1}(L_{s+1}(z+1))$;

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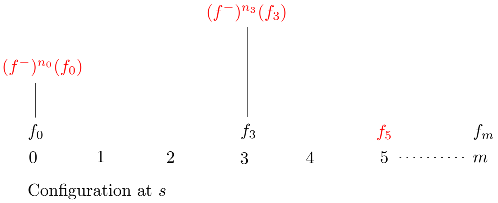
(notice that, in case of conflict between c and c^-, the system considers only c)

Neither c, nor c^{-}: Just go on!



Provisional theses

$A_s = H_s(L_s(m)) = \{(f^{-})^{n_0}(f_0), (f^{-})^{n_3}(f_3), f_5, \ldots\}$

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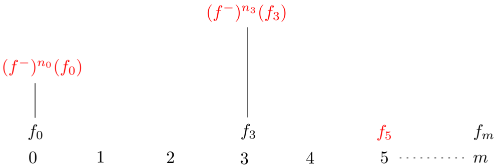
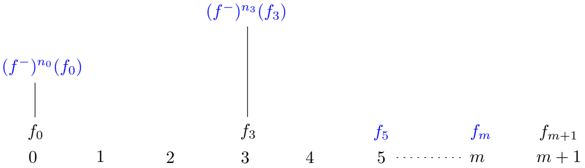


Configuration at s



Configuration at $s+1$, via clause (1)

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$A_s = H_s(L_s(m)) = \{(f^-)^{n_0}(f_0), (f^-)^{n_3}(f_3), f_5, \dots\}$
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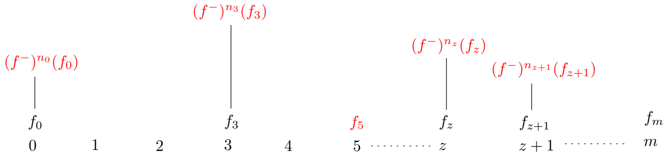
Has c appeared? Then substitute and again!



Configuration at s

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$$A_s = H_s(L_s(m)) = \{(f^-)^{n_0}(f_0), (f^-)^{n_3}(f_3), f_5, \ldots, (f^-)^{n_z}(f_z), (f^-)^{n_{z+1}}(f_{z+1}), \ldots, f_m\}$$

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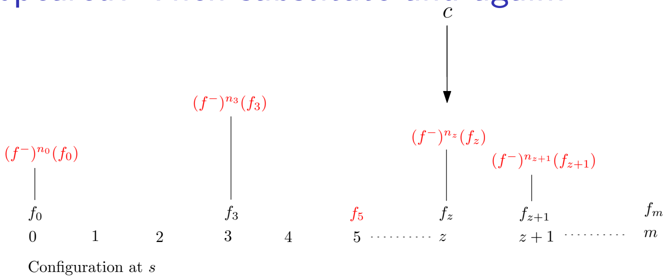

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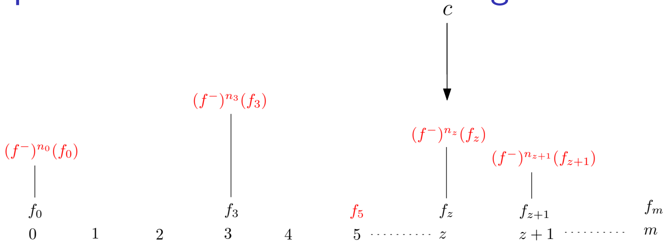
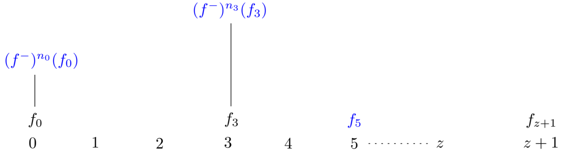


Configuration at s



Configuration at $s+1$, via clause (2)

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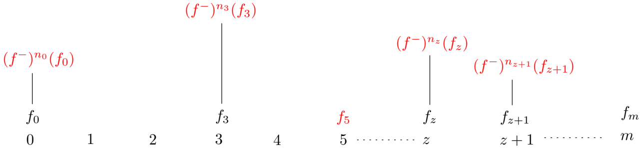
Has c^- appeared? Then substitute and again!



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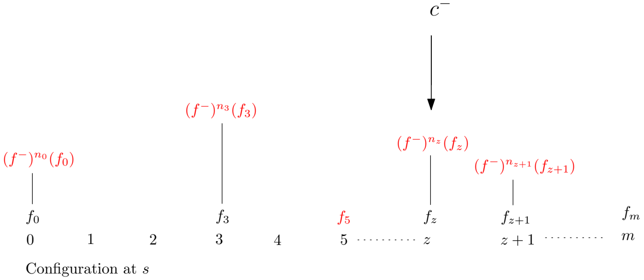

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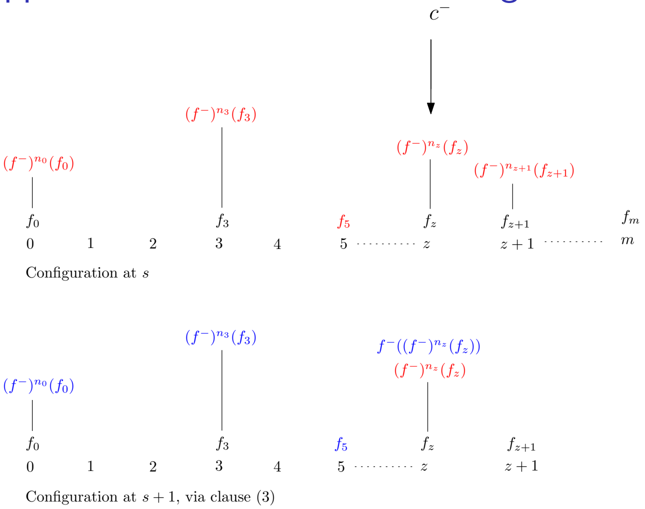

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Final theses and quasidialectical sets

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Definition

A set Q is called quasidialectical if there is a quasidialectical system q such that $A = A_q^\alpha$, for some quasidialectical system q, and for some approximation α to H.

The dependence of the final theses from the approximations

A major difference with respect to dialectical systems is that the set of final theses depends now on which computable approximation to the enumeration operator one chooses.

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Let us show this with an example.

Example of q dependent on the approximations

Consider the quasi-dialectical system $q = \langle H, f, f^{-}, c, c^{-} \rangle$, where $f_x = x$, $f^{-}(x) = x + 2$, $c = 1$, $c^{-} = 2$, and

$$H = \{\langle y, \{2x+1\}\rangle : x, y \in \omega\} \cup \{\langle 0, \emptyset\rangle\} \cup \{\langle y, \{y\}\rangle : y \in \omega\}.$$

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- on the contrary, in β for every x, the axiom $\langle c, \{2x+1\}\rangle$ comes before $\langle c^-, \{2x+1\}\rangle$, so that the third case of the definition of the quasi-dialectical procedure would be used.

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It is easy to see that H is a closure operator, and that there exist computable approximations α and β to H, such that

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- on the contrary, in β for every x, the axiom $\langle c, \{2x+1\}\rangle$ comes before $\langle c^-, \{2x+1\}\rangle$, so that the third case of the definition of the quasi-dialectical procedure would be used.

α and β give rise to different quasi-dialectical sets: $A_q^\alpha = \{0\}$, whereas, for instance $4 \in B_q^\beta$. Moreover, functions $r_s^\alpha(x)$, $\rho_s^\alpha(x)$ have different "asymptotic" behavior from $r_s^\beta(x)$, $\rho_s^\beta(x)$ yielded by β; in particular, we have that $\{\rho_s^\alpha(1) : s \in \omega\}$ is infinite!

Approximated quasidialectical systems

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Definition

An approximated quasidialectical system is a pair (q, α) where q is a quasidialectical system $q = \langle H, f, f^-, c, c^- \rangle$, and α is a computable approximation to H.

Quasidialectical systems with loops

The fact that approximated quasidialectical systems as (q, α) of the last example do exist is not just a matter of curiosity. Indeed, it shows that an approximated quasidialectical system might fail to propose all the axioms. In order to characterize such cases, consider the following definition:

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Definition

Let (q, α) be an approximated quasidialectical system, and y be a slot. We say that (q, α) has a loop over y if $\{\rho_s(y) : s \in \omega\}$ is infinite. If (q, α) has no loops, we call it loopless.

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Therefore, a loop can be visualized as expressing an infinite ascending stack of substitutions over some slot.

Interpretation of loops

To fit loops in our intuitive interpretation is not completely straightforward. Recall Magari's idea of dialectical systems as representing the behavior of a mathematician – or even of a mathematical community – while facing possible contradictions. According to this scenario, quasidialectical systems with loops would describe a mathematical community in which the overall progression of the theory is indeterminately interrupted by a never-ending refinement of a single axiom – a kind of behavior that might be jokingly compared with Kafkian bureaucracy.

Interpretation of loops, continued

However, loops are not so pathological within the theory of quasidialectical systems:

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Moral of the story: not all bureaucracy is pointless!

Characterizing quasidialectical systems with loops

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Lemma (stability)

Let (q, α) be an approximated quasi-dialectical system, and y a slot. If for each $x \leq y$, the pair (q, α) has no loop over x, then $\lim_s r_s(y)$ exists, i.e. there is a stage t such that, for every $s \geq t$, $r_s(y) = r_t(y)$.

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Intuitively, this last result might be understood as stating that there is no loss of information – in terms of the axioms proposed – in working after the stabilization of a given $L(x)$. Indeed, the result shows that any axiom f_x is proposed after stabilization of $L(x)$.

Characterizing q.s. with loops, continued

Lemma

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Let (q, α) be an approximated quasi-dialectical system with loops. Then A_q^α is a c.e. set.

Proof

Call b the least slot over which the pair (q, α) has a loop. By Stability-Lemma, there must be a stage t such that, for all $s \geq t$, $L_s(b) = L_t(b)$: call $X = L_t(b)$.

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Characterizing q.s. with loops, continued

Recall that a c.e. set is said to be simple, if its complement is infinite, and does not contain any infinite c.e. set. As we can see through the next lemma, simplicity gives us a restraint on the kind of information that can be encoded within a loop.

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Let A be a c.e. set. Then there exists an approximated quasi-dialectical system (q, α) with loops such that $A_q^\alpha = A$ if and only if A is coinfinite and not simple.

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Proof, continued

($\Rightarrow$): Suppose that A is c.e. and there is an approximated quasi-dialectical system (q, α) with loops, and $A_q^\alpha = A$. Let b the least slot such that there is a loop over b.

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Thus, at some stage $s > t$, we would see $c^- \in H_s(X)$, contrary to the fact that $L(b)$ does not change after t.

Characterizing q.s. with loops, continued

The conjunction of the last two lemmas give us the following characterization theorem for quasi-dialectical systems with loops:

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Theorem (A1)

The sets that are representable by approximated quasidialectical systems (q, α) *with loops are exactly the c.e. sets that are coinfinite and not simple.*

Locality for loopless quasidialectical systems

We move to loopless approximated quasidialectical system.

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The next lemma states a sort of locality result: even if a quasidialectical system, by means of the revising function f^-, might heavily modify the order in which axioms are tested, what really counts for an axiom f_x to be a final thesis is whether or not f_x has eventually x among its slots.

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Lemma (locality)

Let (q, α) be a loopless approximated quasi-dialectical system. Then $f_y \in A_q^\alpha$ if and only if

$$(\exists t)(\forall s \geq t)[r_s(y) = \langle f_y \rangle] \text{ (and thus } \rho_s(y) = f_y)$$

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Thus, the expressiveness of a quasi-dialectical system without loops, by which it might propose an axiom several times, ends up with a sort of redundancy: among all possible occurrences of f_x in the list of proposed axioms, what really counts is the one that has been proposed at slot x.

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This is certainly true if (q, α) has loops.If (q, α) is loopless, then by Stability Lemma and Locality Lemma, we have that for every x, $\lim_s r_s(x) = r(x)$ and $\lim_s \rho_s(x) = \rho(x)$ exist.

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$$f_y \in A_q \Leftrightarrow (\exists t)(\forall s \geq t)[f_y \in A_s].$$

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Moreover $\lim_s A_s(f_y)$ exists for every y, as after the stage s_0 at which we propose $r_{s_0}(y) = \langle f_y \rangle$, and each $r(x)$, with $x < y$, has reached limit, once we change $\rho(y)$ we can never go back at any future stage s to $\rho_s(y) = f_y$, by f^- being acyclic.

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2. by comparing the overall computational power of dialectical and quasidialectical systems;
3. and finally by investigating whether dialectical and quasidialectical sets coincide or not.

Any dialectical set is representable by a quasidialectical system

First of al notice that every dialectical system is trivially a quasidialectical system, by taking $c = c^-$, with whatsoever f^-. This can be even improved to requiring $c \neq c^-$ in the definition of a quasidialectical systems (indeed, all results in our papers assume $c = c^-$).

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Theorem

Every dialectical set A such that its complement has at least two elements, is represented by a loopless approximated quasi-dialectical system with $c^{-} \neq c$, and the representation is independent of any computable approximation to the enumeration operator of the quasidialectical system.

Dialectical and quasidialectical degrees

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The following result shows that dialectical systems and quasi-dialectical systems coincide with respect of their computational power. In other words, we have that our notion of revision is already somehow encoded in Magari's original proposal.

Dialectical and quasidialectical degrees, continued

Theorem

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Proof

The proof consists of two steps. We first show that every c.e. Turing degree is a dialectical degree; and then we show that every quasidialectical degree is a c.e. Turing degree. Since every dialectical set is quasidialectical, the claim follows immediately.

Every c.e. degree is dialectical

Lemma

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Proof

This is an immediate consequence of the fact that every Π^0_1 set $A \neq \omega$ is dialectical. Thus, if A is c.e. then $A \equiv_T A^c$, and A^c is dialectical.

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Let us first recall the following facts about Δ_2^0 sets. Given a computable function $g(x, s)$ such that, for every x, $g(x, 0) = 0$, and $\lim_s g(x, s)$ exists, recall that the least modulus function m for g, is the function

$$m(x) = \mu s. (\forall t \geq s)[g(x, t) = g(x, s)].$$

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Notice that if A is a Δ_2^0 set, such that $\chi_A(x) = \lim_s g(x, s)$ (where g is a 0-1 valued computable function) and m is the least modulus function for g, then $A \leq_T m$. On the other hand, if B is the c.e. set

$$B = \{\langle x, s \rangle : (\exists t > s)[g(x, t) \neq g(x, s)]\}$$

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then $B \equiv_T m$. So a least modulus function has always c.e. Turing degree.Therefore, if A is a Δ_2^0 set, $g(x, s)$ is a 0-1 valued computable function such that $\chi_A(x) = \lim_s g(x, s)$, for all x, m is the least modulus function for g, and $m \leq_T A$, it follows that A has c.e. Turing degree.

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$$g(f_x, s) = \begin{cases} 1, & \text{if } f_x \in A_s, \\ 0, & \text{otherwise.} \end{cases}$$

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We now show how, using A_q^α as an oracle, we can compute an upper bound for $m(f_x)$. Since f is a computable permutation, this immediately will yield that $m \leq_T A_q^\alpha$.

Proof, continued

If s_x is a stage such that for every $y < x$, $r_s(y)$ has already reached its limit (with $s_0 = 0$), then by the quasidialectical procedure, $r_s(x)$ can change at a stage $s+1 > s_x$, only if $r_{s+1}(x) = r_s(x)^\frown\langle \rho_{s+1}(x)\rangle$, or if $r_s(x) \neq \langle\ \rangle$ and $r_{s+1}(x) = \langle\ \rangle$. In the latter case, by choice of s_x, for every $t \geq s+1$ we have that $r_{s+1}(x) = \langle\ \rangle$.

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- search for the least stage $s > s_x$ such that either $r_s(x) = \langle\ \rangle$, or $\rho_s(x) \in A^\alpha_q$.

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It follows that, for every $s \geq s_{x+1}$, $g(f_x, s) = g(f_x, s_{x+1})$, and thus $m(f_x) \leq s_{x+1}$.

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The following results state that even confining ourselves to loopless quasidialectical systems, they represent a class of sets, which is much larger than the one that is represented by dialectical systems, thus showing the following corollary:

Corollary (A.)

There are loopless quasidialectical sets that are not dialectical.

In fact, much more can be proved, as shown next.

Ershov hierarchy

Since both dialectical and quasidialectical sets are always Δ^0_2 sets, in order to compare them we need a way of comparing the complexity of Δ^0_2 sets. This is provided by the Ershov hierarcy (in which, intuitively, Δ^0_2 sets are ordered w.r.t. *how many* mistakes we make in our best approximations to them).

Definition

We say that a set A is n-c.e. if there is a computable function $g(x, s)$ such that

1. $\chi_A = \lim_s g(x, s)$, and $g(x, 0) = 0$ (thus
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Definition

A set A is ω-c.e. if there are computable functions $g(x, s)$ and $h(x)$ such that, for every x,

1. $A(x) = \lim_s g(x, s)$ and $g(x, 0) = 0$;
2. $|\{s : g(s + 1) \neq g(s)\}| \leq h(x)$.

Idea of the proof

We first to prove that there are dialectical sets in each of the finite levels of Ershov hierarchy.

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We first to prove that there are dialectical sets in each of the finite levels of Ershov hierarchy.

Then, by diagonalizing over the class of finite levels of Ershov hierarchy, we build a quasidialectical sets that is not dialectical (actually we do more: we show that are quasidialectical sets in each of the infinite levels of Ershov hierarchy).

Further work

Future work

1. Dialectical, quasidialectical systems and logical connectives

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3. Explore connections with Learning theory.

Thank you!

Key References

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