Increasing the expressive power of the Carnap first order modal logic C (Abstract)

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In the present paper, we show how the logic **Cpr** increases the expressive power of the Carnap first order modal logic **C**. We also compare this framework to the classical one, given by hybrid modal logic based on Kripke semantics.

In [4], a joint work with Marcin Mostowski, we investigated the recursive complexity of the quantified version of **C**, a Carnapian extension of **S5**. The aim of Carnap in [3] was to design a modal system for logical necessity. In this spirit, we defined the class of **C**-models as **S5**-models of the form (M, S) such that S is the class of all models with the same universe as M. Let us clarify that contrary to Carnap, we took the liberty of allowing models with domains of arbitrary cardinality. The results obtained are nevertheless easily applicable to the authentically Carnapian version of C, which has exactly the same recursive complexity. Neither this logic nor the one we defined is axiomatizable. Despite this fact, the quantified version of **C** has many interesting features. Within it we are able to discriminate between finite and infinite cardinals. To require that the domain is finite, we can state that necessarily each injective function is surjective, or we can require that necessarily, each transitive and antireflexive relation has a maximal element. The modality operator can be understood as a second order quantification over non logical vocabulary, as it does the job of a second order quantification over the space of all functions or relations definable on the domain of the model M in question. It does not mean, however, that **C** has all the expressive power of second order logic. Indeed, we showed in [4] that Löwenheim-Skolem theorems hold for **C**.

In the present paper we first claim that **C** only has the expressive power of Boolean combinations of Σ^1_1 formulae in the empty vocabulary and first order formulae. Then, we concentrate on a way of increasing the expressive power of **C** that we mentioned in [4], classifying vocabulary expressions into rigid and non rigid, in the sense that the necessity operator would be interpreted as universal quantification over non rigid expressions only. This is the requirement that the interpretations of rigid predicates do not change in the different worlds of one and the same model. This allows us to introduce into the language "descriptive" predicates in Carnap's sense (such as *red*), or "natural kind" predicates in Kripke's sense (such as H_2O), by formally treating them as rigid predicates. Let's call such a logic *Carnap partially rigidified modal logic* (**Cpr**). Here we stress that there is indeed a very big step from **C** to **Cpr**, as it allows one to distinguish between different infinite cardinals. Consider a structure $M = (U, A, f)$ and the sentence φ saying that "f is a bijection between A and $U - A$". Taking A rigid and f non rigid in the statement $\diamond\varphi$, we observe that $M \models \diamond\varphi$ iff A and $U - A$ are of the same cardinality. Once we require that both A and $U - A$ are infinite, the negation of $\diamond\varphi$ has only uncountable models. Moreover, given a degree of infinite, we can construct sentences having only models which domain is at least of this cardinality. A direct consequence of this fact is that the downward Löwenheim-Skolem theorem do not hold anymore for **Cpr**, whereas we proved in [4] that it holds (as well as the upward) for **C**. We obtained it as a byproduct of a theorem stating that each **C**-formula is a Boolean combination of formulae possessing this property, formulae of the form $\Box\phi$, where ϕ is a first order formula (theorem 4.5: elimination of nested modalities for **C**). It is not possible anymore in **Cpr** to eliminate nested modalities by the same argument. We concentrate here on the step made when one restricts the class of **S5**-models to consider only **C**-models and we explicate how passing to **Cpr** (that is, removing in **C**-models those worlds in which the interpretations of rigid predicates differ from the ones we want to fix), one gains the possibility of speaking of the cardinality of all the subsets of the domain. Two extreme cases are **Cpr** with only rigid predicates, which is reducible to first order logic, and **Cpr** with only non rigid predicates, which is **C** logic. We study the possible combinations of rigid and non rigid predicates, which are the real means of increasing the expressive power of **C**. We show that **Cpr**'s expressive power is bounded by the first order closure of Σ^1_1 in the full vocabulary. We finally propose a refinement and generalization of the **Cpr** idea, by introducing **Crf** (*Carnap rigidity friendly modal logic*) a logic which can simulate the expressive power of full second order logic. This is formally done by introducing a family of modal operators, each one being indexed by predicates escaping its scope.

Then, we make a comparison between partial rigidification of a modal logic and hybridization of this same logic. We first note that it is possible to rigidify a predicate in some quantified hybrid logics. For that purpose, we mimic the way constants are rigidified in such logics. As suggested by Blackburn in [2], we can express that a constant is rigid in models in which there is a path between every two worlds. We can then say in **S5** that a constant c designates the same individual in every world in M if and only if:

$$M \models \downarrow s.\ \Box \downarrow t.\ (@_s c = @_t c)$$

In the same spirit we can say in **S5** that a predicate A has the same interpretation in every world in M if and only if:

$$M \models \downarrow s.\ \forall x\ \Box \downarrow t.\ (@_s Ax \leftrightarrow @_t Ax)$$

We then propose **QHCL**, an hybridization of **C**, taking large inspiration from [1]. It is not possible to rigidify predicates in this logic, but we can *use* predicates in a rigid way. As for **Crf**, this allows us to simulate the expressive power of full second order logic. In the different extensions of **C**, we discriminate between infinite cardinals by rigidifying predicates which may express such things as "x is a natural number". We compare these formal sentences to natural language sentences such as the following, given by Blackburn (here, one could conceive of the predicate *hip* as "partially rigified") :

Holland will have a hip queen according to nowadays standards.

he proposed to represent it in quantified hybrid temporal logic in the following way:

$$\downarrow s. < F > \downarrow t.\ @_s Hip(@_t q)$$

Finally, we stress that even if some quantified hybrid logics based on Kripke semantics (as **S5**) allow rigidifying predicates, they are much more simpler than the logic **C** and its extensions. In fact, they inherit from the propositional case all general completeness and interpolation results, whereas we proved in [4] that even **Cpr** has exactly the Turing degree of second order logic.

References

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