# The Intended Model of Arithmetic. An Argument from Tennenbaum's Theorem

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It is a well-known fact that first order Peano's arithmetic has infinitely many (continuum) different models. Most of them are called *non-standard* and only one class of isomorphic models is considered as *standard*. (Of course, we count here models of PA up to isomorphism.) We call a model of arithmetic standard if its ordering is of the type $\omega$. We used to consider the standard model of arithmetic as the one that reflects our intuitions about natural numbers adequately. A model that reflects our intuitions adequately we call *intended*.

In this paper we want to answer the following questions: why is one of the interpretations of Peano's axioms distinguished among so many others? Are standard models really intended models?

It is important to notice, that we make a distinction between an *intended* model and a *standard* model of arithmetic. The second notion is well known in metamathematic of arithmetic. The intended model is a model that satisfies intuitions concerning natural numbers. These two notions were often identified. In what follows, we postulate a restriction of the class of intended models to a subclass of standard models.

Benacerraf in [1, 2] presented an analysis of the notion of natural numbers. He argued that any set of objects with the $\omega$–type ordering can be a model for arithmetic. He claimed that it is not important which objects play the role of natural numbers. Important are relations between these objects. This standpoint was further-taken by structuralism. Indeed, as long as we are interested in the question: what is true about natural numbers? it is sufficient to consider any model from the class of isomorphic models with $\omega$–type ordering, since isomorphic models have the same theory.

Our starting point is different. We do not take the property of $\omega$–type ordering as constituting natural numbers. We consider that the basic feature of natural numbers is that we can count using them. Similarly to structuralists we are not interested in questions like: "what is 1?" Contrary to them, we consider that the basic property of natural numbers is the possibility to use them to count. We think that this property plays the decisive role in deciding whether a given model for arithmetic can be considered as being intended or not.

We learn what are natural numbers while learning to count. Consequently, we argue that an intended model for arithmetic should be such that one can perform basic arithmetical operations (addition and multiplication) on elements of this models (numbers from this model). In aim to give the mathematical meaning to the last sentence we use the psychological version of Church's thesis.

**Thesis 1 (The psychological version of Church's thesis)** (*See [4]) Any property that human can compute can be also computed by Turing machines.*

This thesis gives an upper bound on what we may compute. Notice also that from this assumption it follows that in aim to distinguish a class of intended models we cannot identify any two isomorphic models, but at most recursively isomorphic models.

We want to stress here that we treat computability as a basic notion. Moreover, we do not restrict computability to a one fixed domain of objects. We think rather about computability as about an activity which may be related to any set of objects satisfying some basic assumptions. Such an approach may be found for example in Shönfield's book [5].

Church's Thesis together with our basic requirement on computability of arithmetical operations results in the postulate that an intended model for arithmetic has to be recursive.

Our second basic assumption is that any model for arithmetic has to satisfy first order induction. Induction together with Tennenbaum's theorem allows us to describe the class of intended models for arithmetic.

**Theorem 2 (Tennenbaum)** (*See [3]) Let $M$ be a model of Peano arithmetic. If the interpretation of addition and multiplication in $M$ are recursive then $M$ is a standard model for arithmetic (a model with $\omega$–type ordering).*

Now, if we agree that the intended model should be recursive and should satisfy first order induction, then Tennenbaum's theorem tells us, that it has to have the $\omega$–type ordering. Let us notice, that the $\omega$–type ordering of the intended model is in our paper the conclusion and not the starting point of the reasoning.

Our three assumptions: computability of basic arithmetical operations, the psychological version of Church's thesis and the principle of induction together with Tennenbaum's theorem result in our main postulate: the intended model for arithmetic is a recursive model with $\omega$–type ordering. This defines a proper subclass of the standard models for arithmetic that we call intended models. Moreover, it can be easily seen that any two models in this class are recursively isomorphic (the isomorphism function is computable).

## References


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[5] Joseph R. Shoenfield. *Degrees of Unsolvability*. North–Holland, 1971.